GLOBAL UNIQUENESS OF TRANSONIC SHOCKS IN DIVERGENT NOZZLES FOR STEADY POTENTIAL FLOWS

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Abstract. We show that for steady compressible potential flow in a class of straight divergent nozzles with arbitrary cross-section, if the flow is supersonic and spherically symmetric at the entry, and the given pressure (velocity) is appropriately large (small) and also spherically symmetric at the exit, then there exists uniquely one transonic shock in the nozzle. In addition, the shock-front and the supersonic flow ahead of it, as well as the subsonic flow behind of it, are all spherically symmetric. This is a global uniqueness result of free boundary problems of elliptic–hyperbolic mixed type equations. The proof depends on the maximum principles and judicious choices of comparison functions.

1. Introduction and Main Results

This paper is devoted to establishing uniqueness in the large for a class of transonic potential flows with shocks in three–dimensional divergent nozzles. These transonic shocks are spherically symmetric and widely used in aerodynamics and computational fluid dynamics to simulate those transonic shocks appeared in the so called de Laval nozzles (i.e., convergent–divergent nozzles, cf. [1, 8, 10, 11, 15]).

To formulate the problem, it would be somewhat convenient to use (r, θ, ϕ), the spherical coordinates of a point in $\mathbf{R}^3$, with $0 < \theta < \pi, 0 < \phi < 2\pi$. Let Σ be a C^2 domain on the unit 2-sphere $\mathbf{S}^2 \subset \mathbf{R}^3$. For two fixed positive constants $r^0 < r^1$, we call $\Omega = \{(r, \theta, \phi) \,|\, r^0 < r < r^1, (\theta, \phi) \in \Sigma\}$ a straight divergent nozzle, and Σ its cross–section. The wall of the nozzle is the truncated cone $\Gamma = \{(r, \theta, \phi) \,|\, r^0 < r < r^1, (\theta, \phi) \in \partial\Sigma\}$. For $i = 0, 1$, $\Sigma^i = \{(r, \theta, \phi) | r = r^i, (\theta, \phi) \in \bar{\Sigma}\}$ is called the entry and the exit of the nozzle respectively. Then obviously $\partial\Omega = \Gamma \cup \Sigma^0 \cup \Sigma^1$. Recall that the standard Euclidean metric of $\mathbf{R}^3$ written in the local spherical coordinates is $G = dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)$.

We consider steady potential polytropic gas flows in Ω. The governing equations are the following conservation of mass and Bernoulli law (cf. [6, 8]):

$$\operatorname{div}(\rho \operatorname{grad}\varphi) = 0, \tag{1.1}$$

$$\frac{1}{2}\langle \operatorname{grad}\varphi, \operatorname{grad}\varphi\rangle + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} = b_0. \tag{1.2}$$

Since we use spherical coordinates rather than Descartesian coordinates, here div, grad, $\langle\cdot,\cdot\rangle$ are the divergence operator, gradient operator and inner product with respect to the metric G respectively. The unknown φ is the velocity potential (that is, gradφ is the velocity of the flow), b_0 is the known Bernoulli constant determined

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by the incoming flow and/or boundary conditions, ρ is the density, and $\gamma > 1$ is the adiabatic exponent. (Similar to [6], our results and proof also hold for the isothermal case $\gamma = 1$.) The pressure of the flow p and the speed of sound c are determined by

$$p(\rho) = \frac{\rho^\gamma}{\gamma}, \qquad c^2(\rho) = \rho^{\gamma-1}. \tag{1.3}$$

Expressing ρ in terms of $v = \sqrt{\langle \mathrm{grad}\varphi, \mathrm{grad}\varphi\rangle} = \sqrt{(\partial_r\varphi)^2 + \frac{(\partial_\theta\varphi)^2}{r^2} + \frac{(\partial_\phi\varphi)^2}{(r^2\sin^2\theta)}}$:

$$\rho = \rho(v^2) = \Big(1 + (\gamma-1)(b_0 - \frac{1}{2}v^2)\Big)^{\frac{1}{\gamma-1}},$$

equation (1.1) becomes a second order equation for φ:

$$\mathrm{div}(\rho(\langle \mathrm{grad}\varphi, \mathrm{grad}\varphi\rangle)\ \mathrm{grad}\varphi) = 0. \tag{1.4}$$

It is well known that this equation is of mixed elliptic-hyperbolic type in general [8]; it is elliptic if and only if the flow is subsonic, i.e., $v < c$ or equivalently, by the Bernoulli law, $v < c_* := \sqrt{\frac{2}{\gamma+1}\big(1 + (\gamma-1)b_0\big)}$.

Remember that we call Σ^0, Σ^1 the entry and exit of the nozzle respectively. This means that, we always assume:

$$(H) \qquad \partial_r\varphi \geq 0 \quad \text{on} \quad \Sigma^i,\ i = 0, 1.$$

That is, the gas flows in Ω on Σ^0 and flows out of Ω on Σ^1.

Tremendous experiments and numerical simulations have shown that, for a given supersonic flow near the entry of the nozzle, then by giving an appropriately large back pressure at the exit, a transonic shock must appear in the nozzle (see §147 in [8] or §4.3.4 in [11]): the flow is discontinuous across a shock-front; the flow ahead of the shock-front is supersonic, and behind of it is subsonic, with pressure increases (velocity decreases) across the shock-front. The position of the shock-front depends continuously upon the back pressure and vice versa. Therefore, to design nozzles work for special purposes in aerodynamics, one has to understand the existence, uniqueness, and stability of these transonic shocks via rigorous theoretical analysis.

A basic strategy to attack this problem is as follows: (a) formulate a physically meaningful boundary value problem for the governing partial differential equations (PDE), and look for some special solutions involving transonic shocks; (b) study the stability of these special solutions under perturbations of the upcoming supersonic flow, the shape of the nozzle, and the back pressure etc.; (c) study uniqueness of the special solutions in a large class of functions. In step (a), instead of studying the full Euler system, one always constructs some simplified models (such as the potential flow equation, the popular quasi-one-dimensional model of nozzle flow, see [8, 15, 18] and references therein), and then might obtain solutions with particular symmetry by solving the reduced ordinary differential equations or algebraic equations. Step (b) essentially involves nonlinear small-perturbation problems: for example, one solves various linear PDE and then applying some elegant nonlinear iteration techniques to show existence of the nonlinear problem, see, for instance, [2, 3, 4, 5, 7, 10, 12, 16, 17]. In general, it is felt that step (c) is more harder, which involves nonlinear problems which are not of small-perturbation type.

This paper devotes exactly to establishing the uniqueness in the large of a class of spherical symmetric transonic shocks in the divergent nozzle Ω. Such spherical

transonic shocks were constructed in [8] and then widely used to explain the transonic shock phenomena in de Laval nozzles (cf. [8, 11, 15]). Their stability, for the full steady Euler system, in two-dimensional case, was studied by [10] recently. See also [13, 14]. These results provide strong theoretical supports for the applications of these special transonic shocks in practices of aerodynamics, and in computational fluid dynamics to test various numerical schemes designed to capture shocks (cf. [1, 8, 11]).

We remark that in [6], Chen and Yuan have proved uniqueness of a class of flat transonic shocks in straight ducts, and as a byproduct, demonstrated that these flat shocks are instable, therefore not physical. This result is also consistent with previous instability results obtained in the papers [7, 12, 16], which were devoted to the stability issues by studying small-perturbation problems.

Now let us formulate the boundary value problem of the potential flow equation in Ω and summarize some important properties of the special spherically symmetric transonic shocks. Then we will state our main result. The proof is left in the next section.

We suppose that the flow is spherically symmetric and supersonic (i.e., $v > c_*$) on Σ^0; spherically symmetric and subsonic (i.e., $v < c_*$) on Σ^1. More specifically, for a constant $u^0 \in (c_*, \sqrt{2(b_0 + \frac{1}{\gamma-1})})$ and a constant $v_1 \in (0, c_*)$, we consider the following problem:

$$
\begin{aligned}
&(1.4) && \text{in } \ \Omega, && (1.5)\\
&\varphi = u^0 r^0, \quad \partial_r\varphi = u^0 && \text{on } \ \Sigma^0, && (1.6)\\
&\langle \mathrm{grad}\varphi, \mathrm{grad}\varphi\rangle = v_1^2 && \text{on } \ \Sigma^1, && (1.7)\\
&\langle \mathrm{grad}\varphi, N\rangle = 0 && \text{on } \ \Gamma, && (1.8)
\end{aligned}
$$

where N is the outward unit normal of Γ. (If the outward unit normal of $\partial\Sigma$ is (n_1, n_2), then $N = (0, n_1/r^2, n_2/r^2)$.)

We remark that this problem is physically reasonable. In fact, since our purpose is to study transonic shock phenomena in Ω, the flow should be supersonic at the entry, and the equation (1.4) is hyperbolic there in the r-direction. So we need two initial-value conditions like (1.6). We choose φ as in the first condition just to write the solution neatly. The flow is supposed to be subsonic at the exit, where (1.4) is of elliptic type, hence one and only one boundary condition is necessary. Bernoulli type condition (1.7) means that the speed of the flow is given at the exit, and by Bernoulli law, this is equivalent to impose a uniform back pressure at the exit, which is more physical than other conditions, such as φ itself (a Dirichlet condition) (cf. [8]). In addition, (1.8) is the well known impenetrability or slip condition of inviscid flow along solid boundary.

We will study uniqueness of solutions of (1.5)–(1.8) with the following structure (see [6]).

Definition 1.1. For a C^1 function $r = f(\theta, \phi)$ defined on $\bar{\Sigma}$, let

$$
\begin{aligned}
S &= \{(f(\theta,\varphi), \theta, \varphi) \in \bar{\Omega} \,|\, (\theta,\varphi) \in \bar{\Sigma}\},\\
\Omega^- &= \{(r,\theta,\varphi) \in \Omega \,|\, r < f(\theta,\varphi)\},\\
\Omega^+ &= \{(r,\theta,\varphi) \in \Omega \,|\, r > f(\theta,\varphi)\}.
\end{aligned}
$$

Then $\varphi \in C^{0,1}(\bar{\Omega}) \cap C^2(\overline{\Omega^-}) \cap C^2(\overline{\Omega^+})$ is a *transonic shock solution* of (1.5)–(1.8) if it is supersonic in Ω^- and subsonic in Ω^+, satisfies equation (1.4) in $\Omega^- \cup \Omega^+$

and the boundary conditions (1.6)–(1.8) point-wise, the Rankine-Hugoniot jump condition on S:

$$\rho(\langle \mathrm{grad}\varphi^+, \mathrm{grad}\varphi^+\rangle)\langle \mathrm{grad}\varphi^+, \nu\rangle = \rho(\langle \mathrm{grad}\varphi^-, \mathrm{grad}\varphi^-\rangle)\langle \mathrm{grad}\varphi^-, \nu\rangle, \tag{1.9}$$

and the physical entropy condition on S:

$$\langle \mathrm{grad}\varphi^+, \mathrm{grad}\varphi^+\rangle < \langle \mathrm{grad}\varphi^-, \mathrm{grad}\varphi^-\rangle, \tag{1.10}$$

where ν is the normal vector of S, and $\varphi^+(\varphi^-)$ is the limit value along S of φ restricted in $\overline{\Omega^+}$ $(\overline{\Omega^-})$. The surface S is also called the *shock-front*.

For the existence of special transonic shock solutions and their important properties, we have the following lemma. Without any ambiguity, we will also denote by $\varphi^+(\varphi^-)$ the restriction of φ in $\Omega^+(\Omega^-)$.

Lemma 1.1. *Suppose the solution φ depends only on r. Then for a given u^0, we have the following results:*

(1) *There is one solution $\varphi^-(r) \in C^2(\bar\Omega)$ which is supersonic and solves problem* (1.5) (1.6) (1.8) *in the whole nozzle Ω. In addition, $\partial_r\varphi^-(r)$ is strictly monotonically increasing on (r^0, r^1).*

(2) *There exists a connected open interval $I \subset \mathbf{R}^+$ such that for $v_1 \in I$, there is uniquely one transonic shock solution*

$$\varphi_b(r) = \begin{cases} \varphi^-(r), & r \in [r^0, r_{s'}), \\ \varphi_b^+(r), & r \in [r_{s'}, r^1] \end{cases} \tag{1.11}$$

to problem (1.5)–(1.8), *with $S_b := \{(r,\theta,\phi) \in \Omega \,|\, r = r_{s'} \in (r^0, r^1)\}$ being the shock-front, $\varphi^-(r)$ the supersonic flow obtained in* (1), *and $\varphi_b^+(r)$ a subsonic flow. In addition, it has the following properties:*

(i) *$\partial_r\varphi_b^+(r)$ is strictly monotonically decreasing on $(r_{s'}, r^1)$.*

(ii) *$\varphi^-(r_{s'}) = \varphi_b^+(r_{s'})$ and $\varphi^-(r) > \varphi_b^+(r)$ for $r_{s'} < r \le r^1$;*

(iii) *$\partial_r\varphi^-(r) > \partial_r\varphi_b^+(r)$ for $r_{s'} \le r \le r^1$;*

(iv) *For any fixed $R \in (r^0, r^1]$, $\partial_r\varphi_b^+(R)$ may be regarded as a continuous function of $r_{s'}$, and is strictly monotonically increasing for $r_{s'} \in (r^0, R)$.*

Proof. Let $v = \partial_r\varphi$ and solve ρ_0 from $(u^0)^2/2 + (\rho_0^{\gamma-1} - 1)/(\gamma - 1) = b_0$. Consider the following algebraic equations of $v(r)$ and $\rho(r)$ for $r \in (r^0, r^1)$:

$$r^2\rho v = a_0 := (r^0)^2\rho_0 u^0, \tag{1.12}$$

$$\frac{1}{2}v^2 + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} = b_0. \tag{1.13}$$

The claims in the Lemma can then be shown by elementary calculus (see, for example, [8, 15]). In [15] the computation is carried out for the two–dimensional steady full Euler system, but the results there (i.e., Propositions 1, 3 and Theorem 6) still hold for three-dimensional potential flows (except the asymptotic expressions in Proposition 1 as $r \to \infty$, which we do not need in this paper). Recall that $v = \partial_r\varphi$ here is the velocity of the flow in the r-direction. □

We also have the global uniqueness of supersonic flow φ^- in the whole nozzle Ω, which may be proved by standard energy estimates of nonlinear wave equations.

Lemma 1.2. *For a given u^0, the solution $\varphi^-(r) \in C^2(\bar\Omega)$ which solves problem* (1.5) (1.6) (1.8) *is unique in the class of C^2 functions describing supersonic flows.*

Now we state our main result.

Theorem 1.1. *Under the hypotheses* (H)*, for a given* u^0 *and then any* $v_1 \in I$*, there exists one and only one transonic shock solution to problem* (1.5)–(1.8) *in the sense of Definition 1.1, which is exactly the* $\varphi_b(r)$ *constructed in Lemma 1.1 corresponding to* u^0 *and* v_1*.*

Remark 1.1. This uniqueness result also holds if we consider the whole spherical shell $\Omega' = \{(r,P)\,|\,r^0 < r < r^1, P \in \mathbf{S}^2\}$ instead of the nozzle Ω. The proof is similar and simpler.

Since the supersonic flow φ^- is known, independent of downstream conditions and unique in the large according to Lemma 1.2, to prove Theorem 1.1, we just need to show that any possible shock-front must be S_b and the flow behind of it, φ^+, must be φ_b^+. Therefore this is essentially a uniqueness result of a free boundary problem for a nonlinear second order PDE, with the transonic shock-front S being the free boundary. On the free boundary we have two boundary conditions, namely the Dirichlet condition

$$\varphi^+ = \varphi^-, \quad \text{on } S \tag{1.14}$$

due to the continuity of φ across S (see Definition 1.1), and the Neumann condition (1.9).

In the rest of this paper, Section 2, we will present the proof of Theorem 1.1. It might be a little unexpected to find that the proof is not very hard. It depends on maximum/comparison principles of elliptic equations and some judicious choices of the above constructed special solutions as comparison functions. The ideas are generalizations of those in [6], and we need the deeper properties of the family of special solutions as in Lemma 1.1 to make them work. This fact, however, indicates that our methods might be applicable to a large class of free boundary problems which possess families of special solutions with fine structures.

2. Proof of main result

Let

$$\varphi = \varphi(r,\theta,\phi) = \begin{cases} \varphi^-(r), & r < f(\theta,\phi), \\ \varphi^+(r,\theta,\phi), & r \geq f(\theta,\phi) \end{cases} \tag{2.1}$$

be a transonic shock solution to problem (1.5)–(1.8), with $S = \{(r,\theta,\phi) \in \bar{\Omega}\,|\, r = f(\theta,\phi) \in C^1(\bar{\Sigma}), (\theta,\phi) \in \bar{\Sigma}\}$ being the shock-front. Then ν, the normal of S, pointed from Ω^- to Ω^+, in spherical coordinates, is

$$\nu = \Big(1, -\frac{1}{r^2}\partial_\theta f, -\frac{1}{r^2\sin^2\theta}\partial_\phi f\Big). \tag{2.2}$$

As in [6], by the Neumann condition (1.8) and entropy condition (1.10), we can easily show that

$$\langle \nu, N\rangle = -\frac{1}{r^2}(n_1\partial_\theta f + n_2\partial_\phi f) = 0, \quad \text{at } \; S\cap\Gamma. \tag{2.3}$$

That is, the shock-front is always perpendicular to the wall of the nozzle. In addition, direct computation yields

$$\rho\cdot\langle \mathrm{grad}\varphi, \nu\rangle = \rho\cdot\Big(\partial_r\varphi - \frac{1}{r^2}\partial_\theta\varphi\partial_\theta f - \frac{1}{r^2\sin^2\theta}\partial_\phi\varphi\partial_\phi f\Big). \tag{2.4}$$

We may write the equation (1.4) in non-divergence form as

$$\sum_{i,j=0}^{2} A^{ij}(D\varphi)\partial_{ij}\varphi + B(D\varphi) = 0. \tag{2.5}$$

Here, for simplicity, we set $y^0 = r, y^1 = \theta, y^2 = \phi$, and $\partial_{ij} = \partial_{y^i y^j}$. We do not need to write out the specific expressions of (2.5), but it is essential to note that A^{ij} and B depend only on the first order derivatives of φ. This can be seen from the well known form of (2.5) in Descartesian coordinates (x^1, x^2, x^3):

$$c^2\Delta\varphi - \sum_{i,j=1}^{3} \partial_{x^i}\varphi\partial_{x^j}\varphi\partial_{x^i x^j}\varphi = 0. \tag{2.6}$$

Indeed, we can also use this expression below to show the validity of maximum principles.

Now suppose for the given u^0 and $v_1 \in I$, the special transonic shock solution is φ_b and its shock-front is $\{r = r_s\} \cap \bar{\Omega}$. (See Lemma 1.1.) The proof of Theorem 1.1 is then divided into two cases.

Case 1. We first show that if there holds

$$\min_{\bar{\Sigma}} f \geq r_s, \tag{2.7}$$

then $f \equiv r_s$ and $w := w(r,\theta,\varphi) = \varphi_b^+(r) - \varphi^+(r,\theta,\phi) \equiv 0$.

Let us consider the domain $\Omega_f^+ = \{(r,\theta,\phi) \in \Omega \,|\, r > f(\theta,\phi)\}$. By (2.7), both φ^+ and φ_b^+ are well defined in Ω_f^+. Therefore we may formulate a boundary value problem of w in Ω_f^+ as follows.

First, w solves the following linear PDE:

$$\begin{aligned} & \sum_{i,j=0}^{2} a^{ij}(D\varphi_b^+)\partial_{ij}w + \sum_{i=0}^{2} b^i\partial_i w \\ := \quad & \sum_{i,j=0}^{2} A^{ij}(D\varphi_b^+)\partial_{ij}w \\ & + \Big(\sum_{i,j=0}^{2} \partial_{ij}\varphi^+(A^{ij}(D\varphi_b^+) - A^{ij}(D\varphi^+)) + B(D\varphi_b^+) - B(D\varphi^+)\Big) \\ = \quad & 0. \end{aligned} \tag{2.8}$$

Note that there is no zeroth-order term cw here. Since φ_b^+ is subsonic, this equation is uniformly elliptic, and by our requirements in Definition 1.1, all the coefficients a^{ij}, b^i are bounded. So by the strong maximum principle [9], if w is not a constant, its minimum can only be achieved on the boundary $\partial\Omega_f^+ = \Sigma^1 \cup (\overline{\Omega_f^+} \cap \Gamma) \cup S$.

Second, by (1.7), we have the following boundary condition of w on Σ^1:

$$\langle \mathrm{grad}\varphi_b^+ + \mathrm{grad}\varphi^+, \mathrm{grad}w\rangle = v_1^2 - v_1^2 = 0. \tag{2.9}$$

By assumption (H) and Lemma 1.1,

$$\langle \mathrm{grad}\varphi_b^+ + \mathrm{grad}\varphi^+, (1,0,0)\rangle > 0. \tag{2.10}$$

So (2.9) is a linear oblique derivative condition to w. Similarly, on $\overline{\Omega_f^+} \cap \Gamma$, there is a Neumann condition

$$\langle \mathrm{grad} w, N \rangle = 0. \tag{2.11}$$

Therefore by Hopf boundary point lemma [9], if w is not constant, the minimum of w also can not be achieved on $\Sigma^1 \cup (\overline{\Omega_f^+} \cap \Gamma)$. (For points on $\Sigma^1 \cap (\overline{\Omega_f^+} \cap \Gamma)$, as in [6], we may use a locally even reflection arguments to prove that a minimum can not be attained there, since the two surfaces meet there at a right angle.)

Third, suppose the minimum is achieved at a point $(R, P) \in S$ (i.e., $R = f(P), P \in \bar{\Sigma}$). We note that w satisfies a Dirichlet condition here (cf. (1.14)):

$$w = g(\theta, \phi) := \varphi_b^+(f(\theta, \phi)) - \varphi^-(f(\theta, \phi)) \leq 0. \tag{2.12}$$

The inequality holds due to property (ii) in Lemma 1.1, with $r_{s'} = r_s$.

We first suppose that g admits a minimum at P, which is an interior point of Σ. Then $\partial_\theta g = \partial_\phi g = 0$ at P and, since $\partial_r \varphi^-(r) > \partial_r \varphi_b^+(r)$ by property (iii) in Lemma 1.1 (taking $r_{s'} = r_s$), we also have

$$\partial_\theta f(P) = \partial_\phi f(P) = 0, \tag{2.13}$$

which indicates that $\partial_\theta w(R, P) = \partial_\phi w(R, P) = 0$, and

$$\partial_\theta \varphi^+(R, P) = \partial_\phi \varphi^+(R, P) = 0. \tag{2.14}$$

Indeed, by (1.14) we have $\varphi^+(f(\theta, \phi), \theta, \phi) = \varphi^-(f(\theta, \phi))$, hence, for example, we get $(\partial_r \varphi^+)(\partial_\theta f) + \partial_\theta \varphi^+ = \partial_r \varphi^- \partial_\theta f$, and therefore $\partial_\theta \varphi^+(R, P) = 0$ by (2.13).

Now consider the Neumann condition on S. By (1.9), (2.4), (2.13) and (2.14), there should hold

$$\rho(|\partial_r \varphi^+|^2)\partial_r \varphi^+ = \rho(|\partial_r \varphi^-|^2)\partial_r \varphi^- \tag{2.15}$$

at (R, P). We may solve from this algebraic equation uniquely one $\partial_r \varphi^+ < \partial_r \varphi^-$, which is not less than $\partial_r \varphi_b^+$. (Note that $R \geq r_s$. Here we used property (iv) in Lemma 1.1, with $r_{s'} = r_s$ and $r_{s'} = R$.) Hence we have $\partial_r w = \partial_r \varphi_b^+ - \partial_r \varphi^+ \leq 0$ at $(R, P) \in S$. However, we see that $\nu = (1, 0, 0)$ pointed into Ω_f^+ at (R, P) by (2.13), this is a contradiction to the Hopf boundary point lemma, which asserts that there should hold $\partial_r w > 0$ at (R, P), where w attains its minimum.

For $P \in \partial\Sigma$, since S is perpendicular to Γ, then (2.13) still holds, and the above analysis also works.

Therefore w must be a constant in $\overline{\Omega_f^+}$. Now look at (2.12), we get $\partial_\theta f = \partial_\phi f \equiv 0$ for $(\theta, \phi) \in \bar{\Sigma}$. So f is a constant. If $f = r_s$, then $w = 0$ and the uniqueness is proved.

Now we show that $f > r_s$ is impossible. Note that w is a constant and φ_b^+ depends only on r, so φ^+ depends only on r for $f \leq r \leq r^1$, and therefore

$$\varphi(r, \theta, \phi) = \begin{cases} \varphi^-(r), & r^0 \leq r < f, \\ \varphi^+(r), & f \leq r \leq r^1 \end{cases} \tag{2.16}$$

is a special solution to problem (1.5)–(1.8). By Lemma 1.1 (iv), since $f > r_s$, we must have $v_1 = \partial_r \varphi^+(r^1) > \partial_r \varphi_b^+(r^1) = v_1$, a contradiction as desired.

Case 2. We now turn to the case that

$$R_s := \min_{\bar{\Sigma}} f < r_s. \tag{2.17}$$

We will prove by contradiction that this is impossible.

Let

$$\varphi_B(r) = \begin{cases} \varphi^-(r), & r^0 \le r < R_s, \\ \varphi_B^+(r), & R_s \le r \le r^1 \end{cases} \tag{2.18}$$

be a special transonic shock solution constructed in Lemma 1.1 for which $\{r = R_s\} \cap \bar{\Omega}$ is the shock-front. Set $V_1 = \partial_r \varphi_B^+(r^1)$. Then by property (iv) in Lemma 1.1 (with $R = r^1$, $r_{s'} = R_s$ and then $r_{s'} = r_s$),

$$V_1 < v_1. \tag{2.19}$$

Consider the maximum of the function $w := \varphi_B^+ - \varphi^+$ defined in Ω_f^+. Now w satisfies a linear uniformly elliptic equation similar to (2.8), only with φ_b^+ replaced by φ_B^+. The oblique derivative condition on Σ^1 is now

$$\langle \mathrm{grad}\varphi_B^+ + \mathrm{grad}\varphi^+, \mathrm{grad} w\rangle = V_1^2 - v_1^2 < 0, \tag{2.20}$$

and the Neumann condition on $\overline{\Omega_f^+} \cap \Gamma$ is the same as (2.11). So if w is not a constant, its maximum can only be attained on S, where the Dirichlet condition is

$$w = g(\theta, \phi) := \varphi_B^+(f(\theta, \phi)) - \varphi^-(f(\theta, \phi)) \le 0. \tag{2.21}$$

By the definition of R_s, the maximum, 0, can be achieved at the point $P \in \bar{\Sigma}$ where $f(P) = R_s$. Hence $\partial_\theta f(P) = \partial_\phi f(P) = 0$, as well as $\partial_\theta \varphi^+(R_s, P) = \partial_\phi \varphi^+(R_s, P) = 0$ and $\nu(R_s, P) = (1, 0, 0)$, as shown in Case 1.

By the Rankine-Hugoniot condition (1.9), $\partial_r \varphi^+$ should satisfy

$$\rho(|\partial_r \varphi^+|^2)\partial_r \varphi^+ = \rho(|\partial_r \varphi^-|^2)\partial_r \varphi^- \tag{2.22}$$

at (R_s, P). However, it follows from Lemma 1.1 that $\partial_r \varphi_B^+(R_s)$ is the only solution to this algebraic equation which satisfies the entropy condition. Therefore $\partial_r w = 0$ at (R_s, P), which is a contradiction to the Hopf boundary point lemma from which $\partial_r w$ should be negative at (R_s, P) if w is not constant.

Hence w is a constant and by (2.21), it is zero. Now we obtain, from (2.20), a contradiction. Therefore the Case 2 is impossible.

This finishes the proof of Theorem 1.1.

Remark 2.1. In the proof of Case 1, we can also obtain contradiction if w is not constant by analyzing the maximum as in Case 2. But in Case 2 we are restricted to considering only the maximum of w to obtain contradictions.

Remark 2.2. The same proof also works for the uniqueness of cylindrical transonic shocks for potential flows in two-dimensional straight divergent nozzles.

Remark 2.3. As noted in [6], in the proof we just used the fact from Definition 1.1 that the flow on the right hand side of the shock-front is subsonic (i.e., the entropy condition). We do not need any assumption such as the flow should be subsonic in the whole domain Ω_f^+.

Remark 2.4. The assumption (H) is only used to guarantee the condition (2.10) at the exit to be oblique. Since for the family of special solutions φ_b constructed in Lemma 1.1, $\partial_r \varphi_b^+(r^1)$ is positive and bounded away from zero, say, $\partial_r \varphi_b^+(r^1) > c_0 > 0$, therefore we can relax (H) a little by, for example, requiring only that $\partial_r \varphi \ge -c_0/2$ at the exit.

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